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TESSELLATIONS ASSOCIATED WITH NUMBER SYSTEMS

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Resumen

En este trabajo probamos que son iguales la dimensión Hausdorff y la dimensión B ('box-counting', capacidad, entropía) del contorno E de una tesela del plano proveniente de un sistema numérico. Esta dimensión s es mayor o igual a uno y menor que dos. La medida de Hausdorff de E es positiva en su dimensión.

Palabras clave: Sistema numérico, Teselado.

Abstract

We prove that the Hausdorff dimension and box-counting dimension of the boundary E of a tile corresponding to a number system are equal, less than 2 and not less than 1. The Hausdorff measure of E is positive in its dimension.

Key words: Number systems, Tessellation.

1. An auxiliary result on the Hausdorff dimension. The next Theorem 1 can be proved repeating almost *verbatim* the proof given in Theorem 3.1 of Falconer's book [3] only replacing the functions g_i^{-1} that appear there by new functions f_i. For the sake of completeness we prove Theorem 1 in §3.

Theorem 1. Let E be a non trivial compact set and a and r_0 two positive numbers, $r_0 < 1$, such that for any set $U \subset E$, $0 < |U| := diam\ (U) < r_0$, there exist $V = V(U) \subset E$ and a map f from V onto U that verifies

$$v, w \in V \Rightarrow |f(v) - f(w)| \leq \frac{|U|}{a}|v - w|. \qquad (1)$$

Then, the box dimension $\dim_B(E)$ exists and if $s = \dim_H(E)$ then i) and ii) hold:

i) $H^s(E) \geq a^s$

ii) $s = \dim_B(E)$.

2. A basic result on the boundary of a number tile. In this section we assume the next hypothesis:

H) Let $b \in C$ ($\equiv R^N, N = 2$), $|b| > 1$, be the base of the number system $\{b, D\}$ with $D = \{0, a_1, ..., a_n\} \subset R^N$ its set of ciphers (digits) such that there exists a point lattice $L=[1, g]$ $\{m + ng : m, n \in Z\} \subset R^N$ veri-

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fying $bL \cup D \subset L$ with D a complete set of residues modulo b, (*i.e.*, each point y of L can be written in a unique way as $y = bx + c$, $x \in L$, $c \in D$).

Definitions. $F: = \{z : z = 0.\ c_1c_2...;\ c_i \in D\}$ and $F_t: = t + F$.

H') $\{F_t : t \in L\}$ is a tessellation of R^N, (*i.e.*, $R^N = \cup F_t$, $m(F_u \cap F_v) = 0$ for $u \neq v$).

Theorem 2. If **H)** and **H')** hold then

a) the box dimension of $E: = \partial F$ exists,

b) $s = \dim_H E = \dim_B E$,

c) $H^s(E) > 0$.

d) $1 \le s < N$.

Proof. a), b) and c) will follow from Theorem 1. In fact, suppose $U \subset E$ has diameter $|U| < r_0: = \rho/|b|$ where $2\rho: = \min\{|\lambda|; 0 \neq \lambda \in L\}$. Let k be the positive integer verifying $\rho/|b| \le |U|\,|b|^k < \rho$.

We write $U = \bigcup_{j=1}^{M} U_j$, where each U_j is of the form $U \cap (F_{0.b_1...b_k} \cap F_{\gamma.c_1...c_k})$, $\gamma \in S^0: = \{t \in L : t \neq 0,\ F \cap F_t \neq \varnothing\}$ and $b_i, c_i \in D$ depend on j. Let $g_j(z): = b^k z + t_j$ where each $t_j = -\sum_{i=1}^{k} b_i b^{k-i}$ is a point of the lattice L (this because of $bL \cup D \subset L$). Each similitude g_j maps U_j into E and $|g_j(z) - g_h(z)|$ is either identically 0 or $\ge 2\rho$. Therefore, if the maps are not identical then

$$dist\left(g_j(U), g_h(U)\right) \ge 2\rho - |U|\cdot|b|^k > \rho. \qquad (2)$$

Let $V = \bigcup_{j=1}^{M} V_j$ where $V_j: = g_j(U_j)$ and define $f: V \to U$ by $f(z) = g_j^{-1}(z)$ if $z \in V_j$. Observe that if $V_j \cap V_h \neq \varnothing$ then, by (2), g_j and g_h must be identical. Therefore, f is well defined and onto U. We claim that if $z, w \in V$ then

$$|f(z) - f(w)| \le \frac{|U|}{a}|z - w|, \text{ where } a = \rho/|b|. \qquad (3)$$

This will show that the hypothesis of theorem 1 are fullfilled, so a), b) and c) are true.

Let $z \in V_j$, $w \in V_h$. There are two possibilities:

i) g_j and g_h are identical. Then,

$$|f(z) - f(w)| = |z - w||b|^{-k} \le \frac{|U|}{\rho/|b|}|z - w|.$$

ii) g_j and g_h are not identical. Then, using (2), one gets $|z - w| \ge dist\{V_j, V_h\} > \rho$ and $|f(z) - f(w)| \le |U| = \frac{\rho}{\rho}|U| < \frac{|U|}{\rho}|z - w|$.

Thus, in any case (3) is true with $a = \rho/|b|$.

Let us prove d). $s < N$ is a consequence of c) and the definition of tesselation. On the other hand, F is a compact set with non void interior and E is compact. Any compact set with Hausdorff dimension less than 1 is totally disconnected. If $s < 1$ then the complement E' of E in R^N, $N>1$, is a connected set. A polygonal path in E' from one point in int(F) to a point in ext(F) contains necessarily a point in F with two representations. That is, a point in E, a contradiction, QED.

3. Proof of the auxiliary theorem. To prove Th. 1 we shall deduce that

$$\forall d > 0 \quad H^d(E) < a^d \Rightarrow \overline{\dim}_B(E) < d. \qquad (4)$$

Then i) of Theorem 1 is true if $s = 0$ because of $H^0(E) \ge 1$ and if $s > 0$, it is a consequence of (4) since if $d = s$ one obtains the contradiction $\overline{\dim}_B(E) < s$. Besides, for $p > 0$ and $d = s + p$ we have $0 = H^d(E) < a^d$ and from (4) we obtain $\overline{\dim}_B(E) < d$ and ii) follows, qed.

$H^d(E) < a^d$ implies the existence of a finite family of open sets $\{U_i : i = 1,..., m\}$ such that

$$\forall i |U_i| < \inf\{a/2, r_0\} \text{ and } E \subset \bigcup_1^m U_i, \sum_1^m |U_i|^d < a^d.$$

Then, there exists t, $0 < t < d$, verifying $\sum |U_i|^t < a^t$. Let $q: = \sum\left(|U_i|/a\right)^t < 1$.

We obtain from the hypothesis that $\exists V_i := V(U_i)\, \exists f_i : V_i \xrightarrow[onto]{} U_i$ in such a way that

$\forall v, w \in V_i \quad |f_i(v) - f_i(w)| \leq |U_i|\,|v - w|/a.$

Let $I_k := \{1,\ldots, m\}^k$, $I = \cup I_k$ and define $U_{i_1 \ldots i_k} = f_{i_1} \circ \ldots \circ f_{i_k}(V_{i_k}) \subset U_{i_1}$. Then, with some abuse of notation we get,

$$E \subset \cup f_i(V_i) = \cup f_i(E) \subset \cup\{f_{i_1} \circ \ldots \circ f_{i_k}(E): \{i_1, \ldots, i_k\} \in I_k\} = \cup U_{i_1 \ldots i_k}.$$

Let $x = f_{i_1} \circ \ldots \circ f_{i_k}(u)$, $y = f_{i_1} \circ \ldots \circ f_{i_k}(v)$, $x, y \in U_{i_1 \ldots i_k}$. Thus, $u, v \in V_{i_k}$ and it holds that

$$|x - y| \leq \frac{|U_{i_1}|}{a} \left| f_{i_2} \circ \ldots \circ f_{i_k}(u) - f_{i_2} \circ \ldots \circ f_{i_k}(v) \right| \leq$$

$$\leq \frac{\prod |U_{i_j}|}{a^k} |u - v|.$$

In consequence,

$$|U_{i_1 \ldots i_k}| \leq \frac{\prod |U_{i_j}|}{a^k} |E|.$$

Let $\beta := \inf |U_j|/a$, $0 < r < \inf\{|E|, 1\}$. Given $x \in E\ \exists k\ \exists\ U_{i_1 \ldots i_k}$ such that

$$x \in U_{i_1 \ldots i_k},\ r\beta \leq \left(\prod |U_{i_j}|\right)|E| / a^k < r.$$

In fact, $r\beta < r/2 < r < |E|$; *beginning with* U_γ, $\gamma \in I_1$, *we arrive to a first* k *such that* $\frac{|U_{i_k}|}{a} \frac{\prod_{n=1}^{k-1} |U_{i_n}|}{a^{k-1}} |E| < r, \quad r \leq \frac{\prod_{n=1}^{k-1} |U_{i_n}|}{a^{k-1}} |E|.$ *From the definition of* β *we get now*

$$r\beta \leq \frac{\prod_{n=1}^{k} |U_{i_n}|}{a^k} |E| < r.$$

Let $N(r)$ $(< \infty)$ be the minimum number of sets of (positive) diameter less than r that cover E. It holds that

$$N(r) \leq card\left\{\bigcup_k \left\{\gamma \in I_k : r\beta \leq a^{-k} |U_{\gamma 1}| \ldots |U_{\gamma k}| |E| \right\}\right\} \leq$$

$$\sum_{\Lambda \subset I} (|E|/r\beta)^t \prod \left(|U_{\gamma j}|/a\right)^t \leq$$

$$\sum \left\{ \sum_{I_k} (|E|/r\beta)^t \prod_1^k \left(|U_{\gamma j}|/a\right)^t : k = 1, 2, \ldots \right\} \leq$$

$$\left(\frac{|E|}{r\beta}\right)^t \sum_{k=1}^{\infty} \left(\sum_1^m \left(\frac{|U_n|}{a}\right)^t \right)^k \leq \frac{1}{r^t} \left(\frac{|E|}{\beta}\right)^t \sum q^k =$$

$= Mr^{-t}$. Since M is independent of r we have, for $r \to 0$, $\overline{lim}\ \frac{\log N(r)}{\log 1/r} \leq t$. In consequence, $\overline{\dim}_B(E) < d$, QED.

4. Remarks. a) The general context in which these results fit can be seen in [1]

b) In [5] Th. 4 we make more precise the statement a) of Th. 2. There we prove, among other results, that $\dim_B E = \frac{\log \lambda}{\log |b|}$ where λ $(\geq |b|$ because $s \geq 1)$ is the spectral radius of a nonnegative matrix Q and an eigenvalue of maximum modulus of it. Q is in a natural way associated with the system (b, D). With relation to this result the reader may consult [2] and [4].

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