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MANY-VALUED LOGIC: FROM FOUNDATIONS TO APPLICATIONS

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Abstract

We survey some recent applications of the infinite-valued Łukasiewicz calculus in computer science, with particular reference to fault-tolerant search and learning. We also discuss the relations between the algebras of the infinite-valued Łukasiewicz calculus, namely, Chang's MV-algebras, and various areas of mathematics, including lattice-ordered abelian groups, fans and toric varieties, and AF algebras of operators.

Key words: Error-correcting codes, Many-valued logic, MV-algebras and lattice-ordered structures.

Introduction

The infinite-valued propositional calculus of Łukasiewicz today finds interesting applications in computer science, notably in the treatment of uncertain information. Further, its associated algebras, Chang's MV-algebras, have deep relations with various mathematical objects, such as toric varieties, abelian lattice-ordered groups, and the operator algebras of quantum spin systems.

For all unexplained notions and results we refer to Professor Roberto Cignoli's comprehensive overview [24] for these Anales. While in a sense this paper is a continuation of his, we shall mainly focus attention on some recent applications to error-correcting codes and computational learning theory.

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1. Coding and Learning

1.1 *Ulam-Rényi Games*

The ability to guess new, non-casual connections between events is among the main characteristic features of homo sapiens [8]. To understand the simplest aspects of this guesswork one may construct a mathematical model. A notable example is described by Rényi [90, page 47] as follows:

[...] I made up the following version, which I called "Bar-kochba with lies". Assume that the number of questions which can be asked to figure out the "something" being thought of is fixed and the one who answers is allowed to lie a certain number of times. The questioner, of course, doesn't know which answer is true and which is not. Moreover the one answering is not required to lie as many times as is allowed.

For example, when only two things can be thought of and only one lie is allowed, then 3 questions are needed [...]. If there are four things to choose from and one lie is allowed, then five questions are needed. If two or more lies are allowed, then the calculation of the minimum number of questions is quite complicated [...]. It does seem to be a very profound problem [...].

Ulam [104, page 281] essentially poses the same problem:

Someone thinks of a number between one and one million (which is just less than 2^{20}). Another person is allowed to ask up to twenty questions, to each of which the first person is supposed to answer only yes or no. Obviously the number can be guessed by asking first: Is the number in the first half million? then again reduce the reservoir of numbers in the next question by one-half, and so on. Finally the number is obtained in less than $\log_2 (1000000)$. Now suppose one were allowed to lie once or twice, then how many questions would one need to get the right answer?

Both Rényi and Ulam are concerned with the variant of the familiar game of Twenty Questions where two players, Carole and Paul, fix a finite search space, Carole chooses a secret number, and Paul must guess it by asking a minimum number of yes-no questions, with the proviso that Carole can give up to e wrong/inaccurate/mendacious answers.

For concrete applications it is useful to assume that Carole is not aware of giving erroneous answers-that is, she is not lying, but her answers are misunderstood because the transmission channel is noisy. In this way, Carole can be equivalently thought of as an artificial satellite which is transmitting, at each instant $t = 1, 2, 3, \ldots$, a bit b_t carrying the current yes-no answer. Some bit may be occasionally received as $1 - b_t$ instead of b_t, as the result of distortion. Paul's adaptive question then amounts to sending back to Carole a copy of the actually received bit via a noiseless channel[1].

With this formulation, the Ulam-Rényi game becomes a main chapter of Berlekamp's theory of communication with feedback [9] (also see [34]). When the noiseless feedback channel is unavailable, all questions must have been asked non-adaptively at the outset, in a list given in the input of the computer of the satellite: then optimal searching strategies exactly amount to optimal e-error-correcting codes[2].

Friendly and unassuming as they are, Ulam-Rényi games also combine several basic ingredients of learning: Paul's questions are adaptive, are formulated in some language, he must learn as quickly as possible, and his search is fault-tolerant. The mutual interplay between adaptivity, fault-tolerance, efficiency and formal expressive power can be rigorously analyzed in the context of Ulam-Rényi games. In the first part of this paper we shall survey a number of results, and relate Ulam-Rényi games to computational learning theory [105], [4], and to the infinite-valued calculus of Łukasiewicz [29], [57], [102, Section IV].

1.2 *Measuring uncertain information*

Suppose Carole and Paul are playing a Ulam-Rényi game with e lies, over a search space S with M elements. Assuming already t questions have been answered, for each $i = 0, \ldots, e$, let A_i denote the set of those elements of S falsifying exactly i answers. We say that $(A_0, A_1, \ldots, A_e)$ is Paul's *state* of knowledge. Let $x_i = |A_i|$ denote the num-

[1] Indeed, Carole and Paul now co-operate against distortion, and Carole knows Paul's adaptive questioning strategy. The only missing piece of information needed by Carole to determine Paul's question is the sequence of bits $b^*_1, \ldots, b^*_t$ actually received by Paul ($b^*_i \in \{b_i, 1 - b_i\}$).

[2] Among the surveys devoted to Ulam-Rényi games and their applications, let us quote [22] and [52]. See [99] for a non-technical introduction. See [59] for error-correcting codes.

ber of elements of A_i. Then the state $\sigma = (A_0, A_1, ..., A_e)$ is said to be of *type* $(x_0, x_1, ..., x_e)$. If there is no danger of confusion, we shall freely identify $(A_0, A_1, ..., A_e)$ with $(x_0, ..., x_e)$, and say that the latter, too, is a *state*.

Similarly, a question T will be represented by the $(e + 1)$-tuple $[t_0, ..., t_e]$, where $t_j = |T \cap A_j|$.

Suppose Paul asks question $T = [t_0, ..., t_e]$, being in state $(x_0, ..., x_e)$. Suppose Carole's answer is "yes". Then the resulting state $\sigma^{yes} = (x_0', ..., x_e')$ is given by

$$\begin{cases} x_0' = t_0 \\ x_j' = t_j + (x_{j-1} - t_{j-1}), \end{cases} \qquad (1)$$

for $j = 1, ..., e$. The above formulas express the fact that, for an element $x \in S$ to falsify j answers there are two possibilities: either x satisfies T and falsifies j of the previously received answers, or else, x falsifies T, and also falsifies $j - 1$ of the previous answers. One can similarly define the state σ^{no} arising from Carole's negative answer.

Summing up, given a state $\sigma = (x_0, ..., x_e)$ and a question T, Carole's two possible answers to T determine two states σ^{yes} and σ^{no}. Paul will then ask a next adaptive question and, depending on Carole's answer, he will be left in one of the four possible states

$$\sigma^{yes,yes},\ \sigma^{yes,no},\ \sigma^{no,yes},\ \sigma^{no,no}.$$

By induction, Paul's questions determine a binary tree τ, rooted at σ, as follows: The edges of τ are labelled by Carole's answers. Each node contains Paul's state of knowledge, and in case this state is not final[3], the node also contains the next question to be asked. We say that τ is Paul's *strategy*. We say that the state σ has a *winning strategy of size* τ if there exists a binary tree τ of height t, rooted at σ, with labels as above, whose bottom nodes are final states.

Berlekamp [9] introduced and analyzed the following basic measure of e-fault-tolerant information:

[3] A state $\sigma = (A_0, A_1, ..., A_e)$ is final if $\left|\bigcup_{i=0}^{e} A_i\right| = 1$.

For each $e \geq 0$ and $n \geq 0$, the *nth volume*, $V_n(x_0, ..., x_e)$, of a state $(x_0, ..., x_e)$ is defined by

$$V_n(x_0, x_1, \cdots, x_e) = \sum_{i=0}^{e} x_i \sum_{j=0}^{e-i} \binom{n}{j}.$$

The *character* of $\sigma = (x_0, ..., x_e)$ is defined by

$$\mathrm{ch}(x_0, ..., x_e) = \min\{n \mid V_n(x_0, ..., x_e) \leq 2^n\}.$$

A strategy S for a state σ is said to be *perfect* if S is of size $q = \mathrm{ch}(\sigma)$ and is winning for σ.

The appropriate generalization of the notion of "balanced question" for a game with lies, rests on the following:

Theorem 1.1 [9] *Let $\sigma = (x_0, ..., x_e)$ be a state, $e \geq 0$, T a question, σ^{yes} and σ^{no} the two possible states resulting from s after Carole's answer to T. We then have*

(i) For all integers $n \geq 1$,

$$V_{n-1}(\sigma^{yes}) + V_{n-1}(\sigma^{no}) = V_n(\sigma).$$

(ii) If σ has a winning strategy of size n then $Vn(\sigma) \leq 2^n$.

(iii) For any integer $M \geq 1$, the quantity

$$\mathrm{ch}(M, 0, 0, ..., 0) \quad (e \text{ zeros})$$

is a lower bound for the number of questions needed to find and unknown element in the Ulam-Rényi game on e lies over a search space of cardinality M.

Statement (iii) is a far-reaching generalization of the trivial fact that an unknown m-bit number cannot be guessed with less than m questions in the Twenty Questions game without lies.

1.3 *Learning and Noise*

We shall now present a different scenario: The main characters are no longer Carole and Paul, but a Turing machine $\mathcal{M}$ and a team $S = [1, ..., M]$ of expert meteorologists to supervise $\mathcal{M}$'s learning. There are several rounds, one for each day $t = 1, 2, ...$ During round t every member i of S offers her/his weather forecast $b_{ti} \in [0, 1] = [$*sunny,* *rainy*} for the next day. Under the supervision of the experts in S, $\mathcal{M}$ has the task of learning the art of giving infallible weather predictions. To this purpose, on each day t, all bits $b_{t1}, ..., b_{tM}$ are sent (perhaps in some conveniently "compiled" form) to the input of $\mathcal{M}$, the latter is asked to make its own forecast b_t, and at the end of day $t + 1$, $\mathcal{M}$ is told whether b_t or $1 - b_t$ was the correct forecast.

If all experts of S were incompetent, they could hardly teach $\mathcal{M}$ the art of infallible {*sunny, rainy*} forecasts. So, for definiteness, let us assume that S contains precisely one expert x^* who is guaranteed to make $\leq e$ wrong forecasts during all rounds $t = 1, 2, ...$ Stated otherwise, while all other experts will make more than e wrong forecasts in the long run, this is not the case of x^*: after a preliminary stage when x^* may make up to e mistakes, x^* becomes infallible.

Problem. Under the above conditions, how many wrong forecasts must $\mathcal{M}$ make, before acquiring infallible predicting capabilities?

The following result is proved by reformulating this problem in the context of Ulam-Rényi games:

Theorem 1.2 [20] *Let q be the character of the initial state of knowledge in a Ulam-Rényi game with e lies over the search space $S = \{1, ..., M\}$. Then q wrong guesses suffice to solve the above problem.*

Thus algorithm $\mathcal{M}$ acquires infallible predicting capabilities, by trial, error and emulation, after making no more than $q = \mathrm{ch}(M, 0, 0, ..., 0)$ wrong forecasts. During the first stage, $\mathcal{M}$ applies an appropriate search strategy (mimicking Paul's strategy in the Ulam-Rényi game with e lies over a search space with M elements) to identify the special expert x^*. As soon as x^* is detected, $\mathcal{M}$ is guaranteed to make no more than e' errors, where $e' = e - w$ and w is the number of wrong predictions made by x^* so far. The current number of $\mathcal{M}$'s wrong predictions is $\leq q - e + w$. The first stage is over. Afterwards, $\mathcal{M}$ will just emulate x^*, making a maximum of e' wrong predictions: once the last mistake is made, also the second stage terminates, and $\mathcal{M}$ becomes infallible (together with its supervisor x^*).

The above example naturally fits in Littlestone's of *mistake bound model of learning,* and is also a particular case of Valiant's *PAC learning,* [55], [51], [54], also see [3], [105][4].

1.4 *Adaptivity vs Efficiency*

The mathematical theory of Ulam-Rényi games is very rich, and, by the above discussion, can be applied to investigate the mutual tradeoffs between various basic constituents of the learning process. In this section we survey some of the main results.

Reducing the impact of adaptivity

By contrast with what is known in error-correcting coding theory, perfect strategies do exist in Ulam-Rényi games –even if adaptivity is reduced to its (nonzero) minimum:

Theorem 1.3 *Fix e = 1, 2, ... We then have*

(i) For all integers $m \geq 1$, up to finitely many exceptions, there is a perfect strategy to guess an m-bit number with up to e lies in the answers.

(ii) Perfect strategies still exist, under the stronger assumption that questions occur in only two non-adaptive batches.

Proof. Statement (i) was proved for e = 1, 2, 3, and for the general case, respectively in [87], [32], [84], and [100]. It is a far reaching generalization of the fact that, in the game of Twenty Questions one can guess an m-bit integer using m questions.

For a proof of (ii) see [17, 18, 21].

The third statement (iii) is a consquence of well known negative results

[4] For further information on computational learning, prediction on line and related topics see, e.g., [5], [6], [10], [33], [39], [40], [50], [53], [56], [108], [109].

in error-correcting coding theory (see [59, 103]), recalling that non-adaptive Ulam-Rényi games correspond to error-correcting codes.

Random errors/lies

One can naturally investigate situations where the number of lies (distortions, errors) is proportional to the length of the learning process. Thus one assumes that there is a real value $0 < r < 1$, known to both Paul and Carole, such that if Paul asks n questions then Carole is allowed to tell at most $\lfloor r\, n \rfloor$ many lies. As expected, lies are no longer so malicious as in standard Ulam-Rényi games[5]. As a matter of fact, in their paper [101] Spencer and Winkler proved the following[6].

Theorem 1.4 *For* non-adaptive *search over* $S = \{0, 1, ..., M-1\}$ *we have*

(i) If $r < 174$, *then Paul has a winning strategy with* $\Theta(\log_2 M)$ *questions.*

(ii) If $r = 1/4$, *then Paul has a winning strategy with* $\Theta(M)$ *questions.*

(iii) If $r > 1/4$, *then no winning strategy exists for Paul for any* $M \geq 9r/(r - 1/4)$, *no matter the number of questions.*

For fully adaptive *search over* $S = \{0, 1, ..., M-1\}$ we have

(iv) If $r < 1/3$, *then Paul has a winning strategy with* $\Theta(\log_2 M)$ *questions.*

(v) If $r \geq 1/3$, *then no winning strategy exists for Paul for all* $M \geq 5$, *no matter the number of questions.*

Reducing the power of noise

In the "half-lie" variant of the Ulam-Rényi game one assumes that only negative answers can be mendacious. Equivalently stated, one of the two bits 0 and 1 is always immune from distortion, as is the typical case in optical communication [89, 31].

The following result interestingly shows that the lower bound of Theorem 1.1 (iii) is ineffective for half-lies.

Theorem 1.5 [19] *Consider the game with one half-lie, over the search space of all m-bit numbers.*

(i) For no value of m there is a strategy with $\leq \mathrm{ch}(2^m, 0) - 3$ *questions.*

(ii) For infinitely many values of m, $\mathrm{ch}(2^m, 0) - 2$ *questions suffice.*

(iii) For infinitely many vlaues of m, $\mathrm{ch}(2^m, 0) - 2$ *questions do not suffice.*

(iv) For each $m = 1, 2, ...$ *there is a strategy with* $q = \mathrm{ch}(2^m\ 0) - 1$ *questions.*

Comparison questions

With respect to the perfect strategies of Theorem 1.3, much larger search spaces can be handled, and much faster and simpler guessing algorithms can be devised, once questions are suitably restricted, e.g., to comparison questions, as in Ulam's original formulation. Next come *bicomparison* questions, asking.

"Does x belong to one of the two intervals $[p, q]$ or $[r, s]$?".

For the case $e = 2$, in [83] one can find a proof of the following

Theorem 1.6 *For all integers* $m \geq 1$ *other than 2, there is a perfect strategy to guess an m-bit number with up to two lies in the answers, and bicomparison questions. The results does not hold for comparison questions.*

A crucial ingredient of the proof is that bicomparison questions preserve the *shape* of Paul's current state of knowledge $\sigma = (A_0, A_1, A_2)$. More precisely, the relative distribution of the A_i in the totally ordered search space S can be parameterized by eleven integers. Paul can quickly reflect on his state of knowledge resulting from Carole's answers, and hence, he can quickly decide which next bicomparative question should be asked to detect the secret number as quickly as possible.

[5] See [88] for a further discussion. See [41] for a first probabilistic analysis of betting in Ulam-Rényi games.

[6] We use the notation $\Theta(g(x)) = \{f(x) \mid$ there exist constants $c_1, c_2 > 0$, and n_0 such that $0 \leq c_1 g(x) \leq f(x) \leq c_2 g(x)$, for all $x \geq x_0\}$.

1.5 *The Logic of Ulam-Rényi Games*

There is a natural connection between Ulam-Rényi games, learning and logic: The success of our guesswork is proportional to our ability to handle information flows; information travels in small packets, which we must efficiently "connect", in order to produce increasingly complex "hypotheses" and make suitable "deductions", to obtain new truths from old truths. Noise is a source of complication in this logical process. Deduction in classical logic is not fault-tolerant, but deduction in learning and in Ulam-Rényi games is.

As noted by von Neumann's [107], in connection with a related problem,

[...] The theory of automata, of the digital, all-or-nothing type, [...] is certainly a chapter in formal logic [... which is ...] one of the technically most refractory parts of mathematics [... dealing with ...] rigid, all-or-nothing concepts, and has very little contact with mathematical analysis [...] the technically most successful part of mathematics [...] The logic of automata will differ from the present system of formal logic in two relevant aspects.

1. The actual length [...] of the chains of operations will have to be considered. 2. The operations of logic [...] will all have to be treated by procedures which allow [...] malfunctions with low but non-zero probabilities. All of this will lead to theories which are much less rigidly of all-or-nothing nature than the past and present formal logic [...]

von Neumann (1948), quoted in [109, p. 2]

One can hardly doubt that Carole's answers in a Ulam-Rényi game with e lies are propositions. However, for $e \geq 1$, Carole's answers fail to obey classical logic in several respects:

1. Two opposite answers to the same repeated question in general do not lead Paul to inconsistency.

2. The conjunction of two equal answers, to the same repeated question, gives in general more information than a single answer.

3. Paul's search is not guided by the trivial principle that each answer is either true or false. Rather, his balanced strategy is guided by his state of knowledge σ. When $e = 0$, σ boils down to partitioning S into 2 classes: those $x \in S$ which falsify one answer, and those who don't. When $e > 0$, σ similarly classifies all elements S by means of $e + 2$ truth-values. Thus 2-valued logic is naturally replaced by $(e + 2)$-valued logic.

In more detail, let us first note that, up to inessential rearrangements, Paul's state of knowledge is a function ς assigning to each element y in the search space S one of $e + 2$ (suitably normalized) *truth-values*, as follows:

- $\varsigma(y) = 1$ iff y satisfies all answers
- $\varsigma(y) = 0$ iff y falsifies $e + 1$ answers, or more
- $\varsigma(y) = \dfrac{i}{e+1}$ iff y satisfies all answers, with exactly i exceptions ($i = 1, ..., e$).

In particular, the *initial* state of knowledge is the constant function 1 over S. Using this representation of states of knowledge, the state change law (1) acquires a particularly simple form, as follows:

Suppose Paul, being in state ς, asks question $Q \subseteq S$ and receives from Carole a positive answer. Let $\tau: S \to$ {truth-values} be the state of knowledge arising from this answer only. Thus, as a particular case of the above definition, for each $y \in S$, $\tau(y) = 1$ if y satisfies the answer, and $\tau(y) = e/(e + 1)$ otherwise. Then Paul's new state of knowledge ς' will be given by $\varsigma' = \varsigma \odot \tau$, where the symbol $\odot$ denotes Łukasiewicz conjunction $a \odot b = \max(0, a + b - 1)$ for all $a, b \in [0, 1]$.

It follows that

Proposition 1.7 [68] *After Carole's answers to his questions $Q_1, ..., Q_t$, Pauls's state of knowledge is the Łukasiewicz conjuntion of the t states of knowledge resulting from Carole's individual answers.*

One defines the natural order structure on the abelian semigroup $(S, \odot)_e$ of all states of knowledge by writing $\tau' \leq \tau''$ (read τ' is *sharper* than τ'', or τ'' is *coarser* than τ') iff $\tau'(y) \leq \tau''(y)$ for all $y \in S$. For every state $\tau \in (S, \odot)_e$ there is a coarsest state $\neg\tau \in (S, \odot)_e$ which is *incompatible* with τ, in the sense that $\tau \odot \neg \tau = 0$, with 0 denoting the zero state. Specifically, $\neg\tau = 1 - \tau$, where 1 is the initial state. Using the operations $\neg$ and $\odot$ we can express the natural order between states of knowledge, by writing the equation $\tau \odot \neg\varsigma = 0$ instead of the inequality $\tau \leq \varsigma$.

The involutive abelian monoid $(S, \odot, \neg, 1)_e$ of all *states of knowledge* in Ulam game over S with e lies is our first example of an MV-algebra (see below for the definition)[7].

We say that an equation

$$\phi(x_1, ..., x_n) = \psi(x_1, ..., x_n)$$

is *absolute* iff it is valid whenever the variables x_1 are replaced by arbitrary states of knowledge τ_i in any posible MV-algebra of states $(S, \odot, \neg, 1)_e$ arising from a Ulam game with finite search space S and any arbitrarily fixed number e of lies.

The following result (see [29] for a proof) generalizes the trivial remark that the set of valid equations between states of knowledge in the classical game of Twenty Questions coincides with the set of valid equations for boolean algebras.

Theorem 1.8 *Let the terms ϕ and ψ be obtained from the variables $x_1, ..., x_n$ and the constant* 1 *by a finite number of applications of the operations $\neg$ and $\odot$. Then the following conditions are equivalent for the equation $\phi = \psi$:*

(i) The equation is absolute;

(ii) The bi-implication $\phi \leftrightarrow \psi$ is a tautology in the infinite-valued propositional calculus of Łukasiewicz.

(iii) The equation follows from the associativity and commutativity of $\odot$, together with the equations $x \odot 1 = x$, $x \odot \neg 1 = \neg 1$, $\neg\neg x = x$, and $\neg(\neg x \odot y) \odot y = \neg(\neg y \odot x) \odot x$ (there are the defining equations of MV-algebras*) using substitutions of equals by equals.*

2. Mathematical structures

2.1 *MV-algebras*

Introduced in the late fifties by Chang (see [16] for a short historical presentation), after some years of relative quiescency, today MV-algebras are thoroughly investigated by several research groups. As we have seen, an *MV-algebra* is a set equipped with an associative-commutative operation $\odot$, with a neutral element 1, and with an involutive operation $\neg$ such that $x \odot \neg 1 = \neg 1$ and, characteristically,

$$\neg(\neg x \odot y) \odot y = \neg(\neg y \odot x) \odot x. \quad (2)$$

One also introduces the derived operations $0 = \neg 1$ and $x \oplus y = \neg(\neg x \odot \neg y)$[8]. These equations express some properties of the real unit interval [0, 1] equipped with negation $\neg x = 1 - x$ and the operation $x \odot y = \max(0, x + y - 1)$. For instance, direct inspection shows that the left hand member of equation (2) coincides with the minimum of x and y, whenever $x, y \in [0, 1]$, and hence the equation expresses the commutativity of the min operation.

The first basic theorem on MV-algebras, *Chang's completeness theorem,* [14], [15] is as follows

Theorem 2.1 *An equation follows from the above equations for MV-algebras iff it is valid in* [0, 1].

This is a far reaching generalization of the well known fact that the two element set {0, 1} equipped with the operations of involution and max generates the variety of boolean algebras[9].

[7] MV-algebras, are the algebras of the infinite-valued propositional calculus of Łukasiewicz [57]. For background on Łukasiewicz logic and MV-algebras see [29]. Other useful references are [47], [49], [86], [48] as well as the monograph [28]. See [25] for a compact technical survey of MV-algebras and their neighbours.

[8] The present definition of MV-algebra is obviously equivalent to the usual one, based on the $\oplus$ operation and the zero element.

[9] The literature contains many distinct proofs, notably [23], [85] and [98]-the first published proof of the completeness of the infinite-valued Łukasiewicz calculus. Nevertheless, each of these proofs requires substantial background prerequisites from such disparate areas as first order model theory, linear inequalities, free abelian lattice-ordered groups, toric varieties. To the best of our knowledge the first elementary proof is given in [26].

A second relevant fact is given by the following result[10]

Theorem 2.2 [65] *There exists a natural categorical equivalence* Γ *between MV-algebras and abelian lattice-ordered groups (for short,* ℓ*-groups) with a distinguished strong unit. For any* ℓ*-group G with strong unit u, the functor* Γ *equips the unit interval* [0, *u*] *of G with the operation* $u - x$ *and with truncated addiction* $u \wedge (x + y)$*. For any morphism* $\psi: (G, u) \to (H, v)$ *of* ℓ*-groups with strong unit, the functor* Γ *restricts* ψ *to* [0, *u*].

Thus, sitting inside every MV-algebra *A* there is a unique addition, the addition of its corresponding group. One can then unambiguously say that elements of *A* sum up to 1, or that they are linearly independent, or that a certain product-like operation on *A* distributes over addition. This is a preliminary step for representing every MV-algebra as the limit of its partitions (thus generalizing a well known property of boolean algebras), and for the approximability of ℓ-groups by means of *simplicial* groups (free abelian groups of finite rank, equipped with the product ordering given by the natural order of the additive group of integers). See Theorem 2.6 below and the remark after the final corollary of this paper.

A third major result is *McNaughton's representation theorem*:

Theorem 2.3 [58] *Up to logical equivalence, formulas in n variables coincide with the totality of continuous* [0, 1]*-valued piecewise linear functions f over* $[0, 1]^n$*, where each piece of f is a linear polynomial with integer coefficients.*

This result vastly extends the well known theorem stating that formulas in two-valued logic represent all boolean functions. McNaughton's 1951 proof is non-constructive. In the 1994 paper [71] one can find a direct geometric proof, stressing the role of desingularization, an important concept in the theory of toric varieties. The main ideas of this new proof are sketched in the next section.

2.2 Normal form and non-singular fans

In many-valued logic, as well as in boolean logic, disjunctive normal forms (DNF) play a fundamental role, both for automated deduction, and for a deeper understanding of the algebra of formulas [69], [79], [1]. Let $\psi = \psi(x_1, \ldots, x_n)$ be a formula in the infinite-valued calculus.

Since by Chang's completeness theorem, the variety of MV-algebras is generated by the unit interval [0, 1], a routine construction shows that, as an element of the free *n*-generated MV-algebra, the equivalence class of ψ can be identified with a piecewise linear (continuous) function *p*, each piece having integer coefficients. This is the easy part of McNaughton's theorem. To get the converse direction, assuming the function $p: [0, 1]^n \to [0, 1]$ to be piecewise linear with integer coefficients, one proceeds in several steps as follows.

First step. We partition the *n*-cube $[0, 1]^n$ into a complex $\mathcal{C}$ of convex polyhedra with rational vertices such that, over any such polyhedron, the function *p* is linear. Generalizing the familiar construction for 2-dimensional polyhedra (where one adds a maximal set of diagonals) we can subdivide $\mathcal{C}$ into a simplicial complex *S* without adding new vertices. Using Minkowski's convex body theorem, we can further subdivide *S* into a *unimodular* simplicial complex τ: in other words, for each *n*-simplex *T* in τ writing in homogeneous integer coordinates the vertices of *T*, we obtain an $(n + 1) \times (n + 1)$ integer valued matrix M_T whose determinant is equal to ± 1.

Second step. We now construct the family $H(\tau)$ of Schauder hats of τ. By the *Schauder hat* of τ at vertex *w* we mean the uniquely determined piecewise linear continuous function *h* such that $h(w) = 1/d$ (where *d* is the least common denominator of the homogeneous integer coordinates of *w*), $h(v) = 0$ for every vertex *v* of τ other

[10] See [26, 27] for a self-contained proof.

than w, and h coincides with a linear function h_T over each n-simplex T of τ. As an equivalent reformulation of the unimodularity property, the coefficients of the linear polynomial representing h_T are integers. An easy lemma, first proved by McNaughton, and then simplified by Rose and Rosser, yields a formula ϕ_T representing h_T over T. A routine min-max argument now shows that all Schauder hat functions are representable by formulas, and therefore p can be expressed as a disjunction (sum) of Schauder hat formulas. We naturally regard the formulas representing the functions in H(τ) as the basic constituents of a DNF reduction of p.

Third step. To grasp the connection with toric varieties, upon writing in homogeneous integer coordinates the vertices of each simplex T in τ, we obtain a complex Δ of simplicial cones, also known as a *fan*. As one more equivalent reformulation of the unimodularity of τ we can say that Δ is "regular" or "non-singular". By the well known vocabulary of toric geometry [38, p. 329-330], (smooth) toric varieties correspond to (regular) fans, and hence to (unimodular) simplicial complexes of the above kind. It follows [73] that desingularizing a toric variety amounts to subdividing a complex into a unimodular simplicial complex, precisely as is done to compute DNF reductions of McNaughton functions. Our desingularization algorithms, arising from DNF reduction algorithms in infinite-valued logic, yield tight estimates of the Euler characteristic of desingularizations of low dimensional toric varieties [2]. Further, Panti [85] gives a geometric proof of Chang's completeness theorem using the De Concini-Procesi theorem on elimination of points of indeterminacy [38, p. 252].

For their proof De Concini and Procesi used a special starring procedure (along two-dimensional cones) yielding finer and finer subdivisions. Iterated application of the same procedure now yields:

Theorem 2.4 *Let Δ be a non-singular fan and $|\Delta|$ the set-theoretic union of its cones. Then there is a sequence*

$$\Delta < \Delta_1 < \Delta_2 < \dots \qquad (3)$$

of star subdivisions of Δ such that every piecewise homogeneous linear functions f with integer coefficients over $|\Delta|$ is a Δ_n-linear support function[11]*, in symbols $f \in SF(\Delta_n)$ for some n.*

Corollary 2.5 *The above sequence (3) generates a direct system of simplicial groups and positive homomorphisms φ_i whose limit is isomorphic to the lattice-ordered abelian group G_Δ of all piecewise homogeneous linear functions with integer coefficients over $|\Delta|$.*

2.3 Product, Partitions, Probability

As shown by recent research on "generalized conjunctions" (also known as, T-norms) [13], [49], [97], a substantial portion of the expressive power needed for applications in many-valued logic and many-valued probability theory would be provided by a logic incorporating a *product* connective jointly with Łukasiewicz disjunction and negation. In general, Łukasiewicz conjunction does not satisfy this distributivity law. The tautology problem of any such logic will be significantly harder than for the infinite-valued calculus of Łukasiewicz which is co-NP-complete like its two-valued counterpart [66].

Various authors have considered many-valued logics with product (see, e.g., [64], [37], [7], [78] and [92]. In [77] the author does not define a new logic, but rather investigates tensor products. This is the bare minimum needed for Weierstrass-like (if-then-else) approximations of continuous [0, 1]-valued functions, in terms of disjunctions of products $h_i \otimes a_i$ for a suitable partition h_i of the cube $[0, 1]^n$.

Introduced in [72] and [74], *states* are the MV-algebraic generalization of *finitely additive* probability measures on boolean algebras. States are also used in

[11] i.e., f is linear over each cone of Δ_n and is integer-valued over each integer point of $|\Delta|^n$.

[41] for a probabilistic approach to Ulam game. On the other hand, countably infinitary operations are needed for the development of MV-algebraic measure theory. Accordingly, σ-complete MV-algebras and σ-additive states are systematically used in the book by Riečan and Neubrunn [97]. As shown by Riečan and his School, many important reults of classical probability theory based on σ-complete boolean algebras and σ-fields of sets have interesting MV-algebraic generalizations. One more example can be found in [77].

While the theory of σ-additive MV-algebraic states is fairly well understood, random variables (alias, observables) still lack a definitive systematization in the context of MV-algebras. A number of technical problems, also involving product and infinite distributive laws are posed by the theory of continuous functions of several (joint) MV-algebraic observables. (See [91]-[96] for interesting positive results). A useful tool for understanding such observables is given by the MV-algebraic generalization of the notion of boolean partition [74], [75]. An *MV-partition* in A is a multiset of linearly independent elements of A whose sum equals 1. As noted above, this definition makes sense, by referring to the underlying **Z**-module structure of the unique ℓ-group G with unit 1 given by $\Gamma(G, 1) = A$. The joint refinability of any two MV-algebraic partitions on an MV-algebra A depends on the *ultrasimplicial* property of its associated ℓ-group G, in the sense that every finite set in G^+ is contained in the monoid generated by some *basis* $B \subseteq G^+$, i.e., a set B of positive elements that are independent in the **Z**-module G. When G is countable, an equivalent reformulation of this property is that G is the limit of an ascending sequence of free abelian groups of finite rank with product ordering, and one-one monotone homomorphisms. After some partial results of Elliott, Handelman and others (see [67], [66], [76], [81] and references therein), recently Marra [60] has proved the following

Theorem 2.6 *Evelyn ℓ-group is ultrasimplicial*

For further results concerning the relationships between MV-algebras and partially ordered groups, see [36], [42]-[45] and [61]-[63].

2.4 MV-algebras and AF algebras

AF (approximately finite-dimensional) algebras, are currently used for the mathematical description of infinite quantum spin systems by operator algebras [46], [70]. Despite their form a very small subclass of C^*-algebras they are very interesting mathematical objects.

As is well known, up to isomorphisms, the most general possible finite-dimensional C^*-algebra $\mathcal{F}$ is a finite direct sum $\mathcal{M}_{d(1)} + \mathcal{M}_{d(2)} + \ldots + \mathcal{M}_{d(t)}$ where $\mathcal{M}_{d(i)}$ denotes the C^*-algebra of all $d(i) \times d(i)$ complex matrices, for suitable $1 \leq d(i)$.

An *approximately finite-dimensional* (for short, *AF*) algebra is the norm closure of the union of an ascending sequence $\mathcal{F}_1 \subseteq \mathcal{F}_2 \subseteq \ldots$ of finite-dimensional C^*-algebras, with the same unit.

By a *projection* p in a C^*-algebra $\mathcal{A}$ we mean a self-adjoint idempotent $p = p^* = p^2$. Two projections $p, q \in \mathcal{A}$ are *equivalent* iff there exists $v \in \mathcal{A}$ such that $vv^* = p$ and $v^*v = q$. We denote by $[p]$ the equivalence class of p, and by $L(\mathcal{A})$ the set of equivalence classes of projections of $\mathcal{A}$. The *Murray-von Neumann order* over $L(\mathcal{A})$ is defined by: $[p] \leq [q]$ iff p is equivalent to a projection r such that $rq = r$.

Elliott's partial addition is the partial operation $+$ on $L(\mathcal{A})$ obtained by adding two projections whenever they are orthogonal. Then $+$ is associative, commutative, monotone, and satisfies the following *residuation* property: For every projection $p \in \mathcal{A}$, among all classes $[q]$ such that $[p] + [q] = [1_{\mathcal{A}}]$ there is a smallest one, denoted $\neg[p]$, namely the class $[1_{\mathcal{A}} - p]$.

Building on Elliott's classification theory [35] along with [65], in [80] the authors proved the following

Theorem 2.7 *In every AF algebra $\mathcal{A}$ there is at most one extension of Elliott's partial addition to an associative, commutative, monotone operation $\oplus$ over the whole $L(\mathcal{A})$ having the above residuation property. Such extension $\oplus$ exists iff $L(\mathcal{A})$ is a lattice. Further, letting $x \odot y = \neg(\neg x \oplus \neg y)$, and $\mathcal{K}(\mathcal{A}) = (L(\mathcal{A}), [0], [1_{\mathcal{A}}], \neg, \oplus, \odot)$ the map $\mathcal{A} \mapsto \mathcal{K}(\mathcal{A})$ is a one-one correspondence between isomorphism classes of AF algebras whose $L(\mathcal{A})$ is lattice-ordered, and isomorphism classes of countable MV algebras.*

In particular, the above map sends isomorphisms classes of commutative AF algebras one one onto isomorphism classes of countable Boolean algebras[12].

An AF algebra $\mathcal{A}$, has a lattice-ordered $L(\mathcal{A})$ iff its (Grothendieck) K_0-group is lattice-ordered. Thus the above correspondence is functorial and preserves much of the MV-algebraic structure. For instance, the finitely additive states of any MV-algebra are in one one correspondence with *tracial states* of its corresponding AF algebra [72, 74].

In the converse direction, to see how an AF algebra with lattice-ordered K_0-group can be approximated by its finite-dimensional subalgebras, one can work in a much simpler MV-algebraic set up, or else, transfer to ℓ-groups the Schauder hat machinery developed for MV-algebraic DNF reductions. From the ultrasimplicial property of every ℓ-group C one gets, when G is countable and has a distinguished strong unit u, a sequence $\varphi_i: \mathbf{Z}^{n_i} \to \mathbf{Z}^{n_{i+1}}$ of simplicial groups with strong units, and positive unit preserving homomorphism which, via the K_0 functor, yields an ascending sequence of finite-dimensional subalgebras whose union is dense in the AF algebra corresponding to (G, u). Note that φ_i is a positive matrix with integer entries.

Going backwards through the composite functor $\Gamma \circ K_0$ one can construct from the free MV-algebra F over countably many generators, a "free" AF algebra $\mathcal{U}$, inheriting the universal properties of F. In particular, every AF algebra with lattice-ordered K_0-group is a quotient of $\mathcal{U}$ [65].

Computations of AF algebras

Using the composite functor $\Gamma \circ K_0$, every AF algebra $\mathcal{E}$ with lattice-ordered $K_0(\mathcal{E})$ can be presented as a sequence of strings of symbols –the Lindenbaum algebra of some theory Θ in the infinite-valued calculus. From Θ one can uniquely recover $\mathcal{E}$. The complexity of the decision problem of Θ measures the complexity of $\mathcal{E}$. While, as proved in [66], the tautology problem for the infinite-valued calculus in co-NP complete, many AF algebras in the literature have polynomial time complexity.

Let Θ be a theory (i.e., a deductively closed set of formulas) in the infinite-valued calculus of Łukasiewicz . We say that Θ is *prime* if, for every pair of formulas α, β, either $\alpha \to \beta \in \Theta$, or $\beta \to \alpha \in \Theta$. As shown by Chang [15, Lemma 1], Θ is prime iff its Lindenbaum MV-algebra (whose elements are the propositional formulas modulo Θ) is totally ordered. Let $FORM_n$ denote the set of all formulas whose propositional variables are among $X_1, ..., X_n$. Let $FORM$ denote the set of all formulas. In [82] the following result was proved:

Theorem 2.8 *Fix an integer $n \geq 1$, and let $\Theta \subseteq FORM_n$ be a recursively enumerable prime theory in the infinite-valued calculus of Łukasiewicz. Then Θ is decidable. By contrast, there exists an undecidable recursively enumerable prime theory $\Psi \subseteq FORM$.*

Thus the phenomenon of Gödel incompleteness is impossible in prime infinite-valued theories and, modulo the composite functor $\Gamma \circ K_0$, it is also impossible in AF algebras whose Murray von Neumann order of projections is total.

One can also investigate computability issues on AF algebras using their representations via diagrams. As a preliminary step one can naturally ask for algorithms deciding when two sequences of positive integer matrices φ_i represent isomorphic AF algebras. As shown in [11],

[12] see [30] for other particular cases of the above correspondence.

[12], for those stable AF algebras arising from a constant sequence of integer matrices $\varphi = \varphi_1 = \varphi_2, \ldots$, there is a Turing machine which, having in its input two matrices φ' and φ'' decides in a finite number of steps whether the corresponding stable AF algebras are isomorphic.

As remarked above, every non-singular fan Δ naturally generates a Bratelli diagram, whose matrices correspond to the starring subdivisions in a suitable De Concini-Procesi elimination procedure for points of indeterminacy in (the toric variety corresponding to) Δ. We can then prove the undecidability of the isomorphism problem for the resulting class of stable AF algebras. The problem is equivalent to the isomorphism problem for the Lindenbaum algebras of two finitely axiomatizable theories in the infinite-valued calculus of Łukasiewicz.

Closing a circle of ideas, with reference to Theorem 2.4 and to Corollary 2.5, Elliott's classification now yield

Corollary 2.9 *Regarding the positive homomorphism φ_n as embeddings of finite-dimensional C*-algebras, we get from* (3) *a stable AF algebra $\mathcal{A}_\Delta$ whose Murray von Neumann equivalence classes of projections form a lattice. Further $G_\Delta \cong K_0(\mathcal{A}_\Delta)$.*

Regarding the φ_n as refinement morphisms of MV-partitions, for every choice of a strong unit u in G_Δ we get a countable MV-algebra $M_\Delta = \Gamma(G_\Delta, u)$.

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