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CHAOS IN A SIMPLE DYNAMICAL SYSTEM

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Resumen

Hemos investigado la dinámica de un modelo simple de satélite galáctico mediante los exponentes de Liapunov de sus órbitas, hallando que un alto porcentaje de ellas (24,1%) son caóticas. Todas las órbitas obedecen la integral de Jacobi pero, además, los exponentes de Liapunov revelan la existencia de otras dos integrales (o pseudo integrales) aislantes. El que una órbita dada pueda estar condicionada por una, o las dos, de dichas integrales depende del valor de la integral de Jacobi para dicha órbita.

Palabras clave: Dinámica galáctica, Caos, Satélites galácticos.

Abstract

We investigated the dynamics of a simple model of galactic satellite through the Liapunov exponents of its orbits, and we found that a high percentage (24,1%) of them are chaotic. All the orbits obey the Jacobi integral but, besides, the existence of two additional isolating integrals (or pseudo integrals) is revealed by the Liapunov exponents. Whether an orbit may be limited by one, or both, of those two integrals depends on the value of the Jacobi integral for that orbit.

Key words: Galactic dynamics, Chaos, Galactic satellites.

1. Introduction

The dynamics of the stars that make up a galactic satellite (i.e., a small stellar system in orbit within a larger one) pose an interesting problem and Carpintero et al. [1999 and 2002] and Muzzio et al. [2000a, 2000b and 2001] have found significant chaotic motions in several models of such systems. Most of those investigations classified the orbits by means of the frecuency analysis code of Carpintero and Aguilar [1998] and made only a limited use of the Liapunov exponents [see, e.g., Lichtenberg and Lieberman 1992] to characterize the chaotic motions. While obtaining Liapunov exponents is much more computationally demanding than frecuency analysis, the former provide additional information on the chaotic orbits (alternatively, the latter allows the classification of the regular orbits).

Here we present an example of how

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the Liapunov exponents can help us to gain insight on the chaotic dynamics of galactic satellites.

2. Model and method

We chose the simplest possible model of a galactic satellite placing a Plummer (or Schuster) sphere [Binney and Tremaine, 1987] on a circular orbit inside a galaxy represented by a logarithmic potential [Muzzio et al. 2000b]. The equations of motion and the Jacobi integral are given by Carpintero et al. [1999] for a modified Satoh (rather than Plummer) potential. It suffices to make **g** go to zero and **h** to infinity (with the product **2gh** going to the square of the softening parameter of the Plummer sphere) in their equations to get the corresponding equations for the Plummer sphere. While the model is not very realistic, because galactic satellites are affected by tidal forces and cannot be spherical, it helps to bridge the gap between more realistic models and the three-body problem of celestial mechanics, where chaos had already been reported by Jefferys [1966] and Hénon [1966a and b].

We adopted a gravitational constant G = 1, a satellite mass M = 1 and a softening parameter for the Plummer sphere b = 0.229; the orbit of the satellite has a radius R = 100 and an angular velocity $\omega = 0.5$, resulting in a tidal radius $r_t = 1.26$ for the satellite. Since the orbit is circular, the Jacobi integral (i.e. the energy integral in the rotating system centered on the satellite) holds. We chose the additive constant of the potential so that the potential is zero at the tidal radius (i.e., with this choice the Jacobi integral must be negative to have bound orbits). As in our previous papers, we divide the values of the Jacobi integral by the value of the potential at the center of the satellite, thus getting an adimensional parameter, that we dub reduced energy, to characterize the orbits. The range of the reduced energy goes from 0 (for a star on the verge of becoming unbound) to 1 (for a star resting at the center of the satellite). Note that, as the potential at the center of the satellite is negative, low values of the reduced energy correspond to loosely bound orbits and high values to strongly bound ones (i.e., the reduced energy increases when the Jacobi integral decreases and viceversa).

We computed the Liapunov exponents using the LIAMAG routine, kindly made available to us by D. Pfenniger, which uses the method of Benettin et al. [1980]. It is worth recalling that, since numerical integrations extend over a finite time interval while the Liapunov exponents are defined for an infinite span, we can only obtain approximate values which may differ somewhat from the theoretical ones. For example, since ours is an autonomous Hamiltonian system, two of the six Liapunov exponents must be zero, while the other four are grouped in two pairs whose members have the same absolute value, but opposite signs. One, or both, of those pairs can also have zero value: in the former case (as well as when all four exponents are nonzero) we have a chaotic orbit, while in the latter we have a regular one. The existence of at least one pair of zero Liapunov exponents is due to the Jacobi integral, and each additional pair of zero values reveals the presence of another isolating integral (or pseudo-integral) of motion. Nevertheless, numerically computed Liapunov exponents are never equal to zero: as the integration time increases, they either stabilize near a non-zero value (chaotic cases), or go on decreasing without never reaching zero for finite integration times (regular cases).

We used the distribution function of the Plummer model [see, e.g., Binney and Tremaine 1987] and a random number generator to create initial positions and velocities for the orbits; a cut-off was imposed so that, after placing the satellite on its orbit, no initial condition could fall beyond the zero-value equipotential or yield a positive value of the Jacobi integral. A total of 1,000 initial conditions were generated in this way and the corresponding orbits were integrated over a time interval of 35,000 units with the LIAMAG routine; the renormalization interval was set at 3.5 units.

3. Results

Figures 1, 2 and 3 show our results as a function of the reduced energy: L1, L2 and L3 stand, respectively, for the highest, intermediate and lowest positive values of the computed Liapunov exponents (recall that, in theory, the lowest value should always be zero and that the difference is due to the numerical approximation only). We notice that the three exponents are essentially zero for all the orbits with large reduced energy values, L1 is not zero for many orbits with intermediate energies, and both L1 and L2 are not zero for many orbits with low reduced energy values.

More quantitative results can be obtained as follows. Since, as explained, numerically computed Liapunov exponents are always larger than zero, we can establish a limiting value above which the exponents can be regarded as clearly non-zero and below which they can be taken as bona fide zero values. Figure 3 suggests that such limiting value can be taken as about 0.0005 and further confirmation can be obtained computing the average values and the dispersions of the Liapunov exponents that fall below that limit: they turned out to be 0.000357 ± 0.000001 (σ = ±0.000021), 0.000294 ± 0.000002 (σ = ±0.000047) and 0.000218 ± 0.000002 (σ = ±0.000061), respectively for L1, L2 and L3, so that the adopted limit falls beyond three or four times the dispersion from those mean values, confirming that it is a reasonable choice.

Overall, we find that 24.1% of the orbits have L1 > 0.0005 and can thus be classified as chaotic; moreover, 16.2% have both L1 and L2 larger than 0.0005, so that they are not only chaotic but they have no other isolating integral than the one of Jacobi. There are no Liapunov exponents larger than 0.0005 for reduced energy values in excess of 0.695, and all the cases with L2 > 0.0005 correspond to reduced energy values smaller than 0.382.

Muzzio et al. [2000a] suggested that the origin of the chaotic motions is the interaction of the three forces that are present in the system: the attraction by the satellite itself, the centripetal-centrifugal force and the Coriolis force; the last two are, of course, due to the orbital motion of the satellite within the larger galaxy. In particular, Muzzio et al. [2000a] indicated that the interaction between the attraction from the satellite and the Coriolis force may be the cause of the chaotic motions observed at intermediate reduced energy values, because the velocities (on which the Coriolis force depends) increase towards the center of the satellite and the centripetal-centrifugal force is relevant in its outermost regions only. This idea can be easily checked with our numerical experiments, repeating the computations after having artificially suppressed the terms of the Coriolis force in the LIAMAG routine; since the Jacobi integral does not depend on the Coriolis force, an orbit is characterized by the same reduced energy value irrespective of whether the Coriolis force has been included in the integration or not. Figure 4 presents our results and we notice that, with the Coriolis force suppressed, not a single orbit with reduced energy larger than 0.216 has any non-zero (within the numerical approximation) Liapunov exponent; besides, only some orbits with reduced energies lower than 0.174 have two non-zero Liapunov exponents. Clearly, the Coriolis force has a dominant role in the onset of chaos at intermediate reduced energy values. Alternatively, it seems to have a soothing influence on the chaotic orbits with low reduced energy values, because the Liapunov exponents for those orbits increased by about 50% after we suppressed the Coriolis force from our computations.

4. Discussion

Our results prove that chaotic motions are very significant in this simple model, not only for the large percentage of chaotic orbits present, but also for the large values of the Liapunov exponents we found. The inverses of those exponents give the Liapunov times, which provide the time scales characteristic of the corresponding chaotic processes. Here we have a couple of Liapunov times as short as 11 (L1 = 0.09), and the bulk of them are shorter than 100 (L1 = 0.01) for chaotic orbits; since periods for circular orbits in the isolated Plummer sphere range from 0.7 at its center to 9.1 at the tidal radius, we see that chaotic effects become significant after between about 1 and 100 orbital periods only. These short Liapunov times compare favorably, not only with the age of galactic satellites (of the order of 1,000 in our units), but even with the relaxation times of globular clusters (of the order of 100 in our units). Thus, our simple model shows that chaotic motions may be important enough in galactic satel-

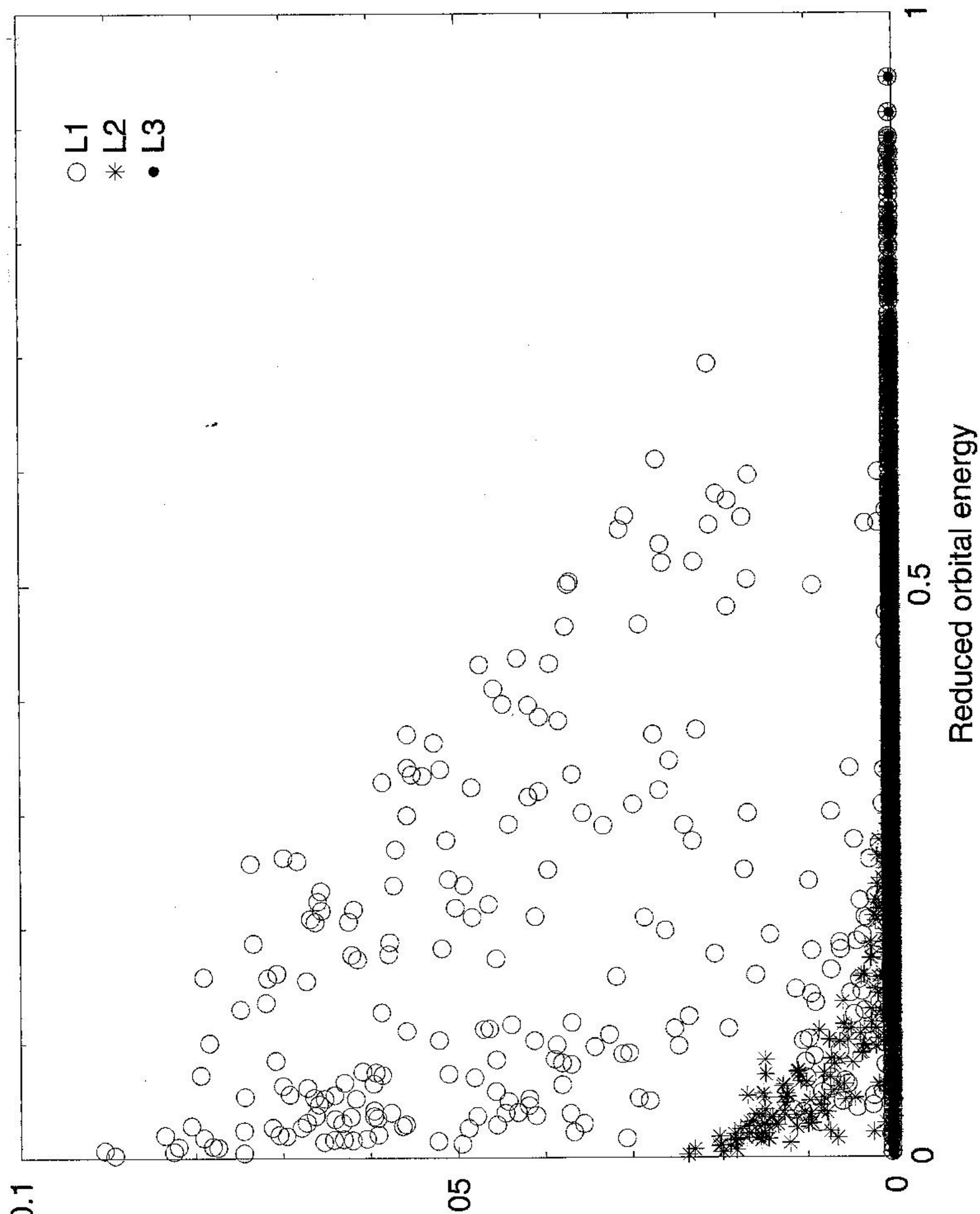

Fig. 1. Computed Liapunov exponents vs. the reduced orbital energy. L1, L2 and L3 stand, respectively, for the largest, the intermediate and the lowest positive Liapunov exponents.

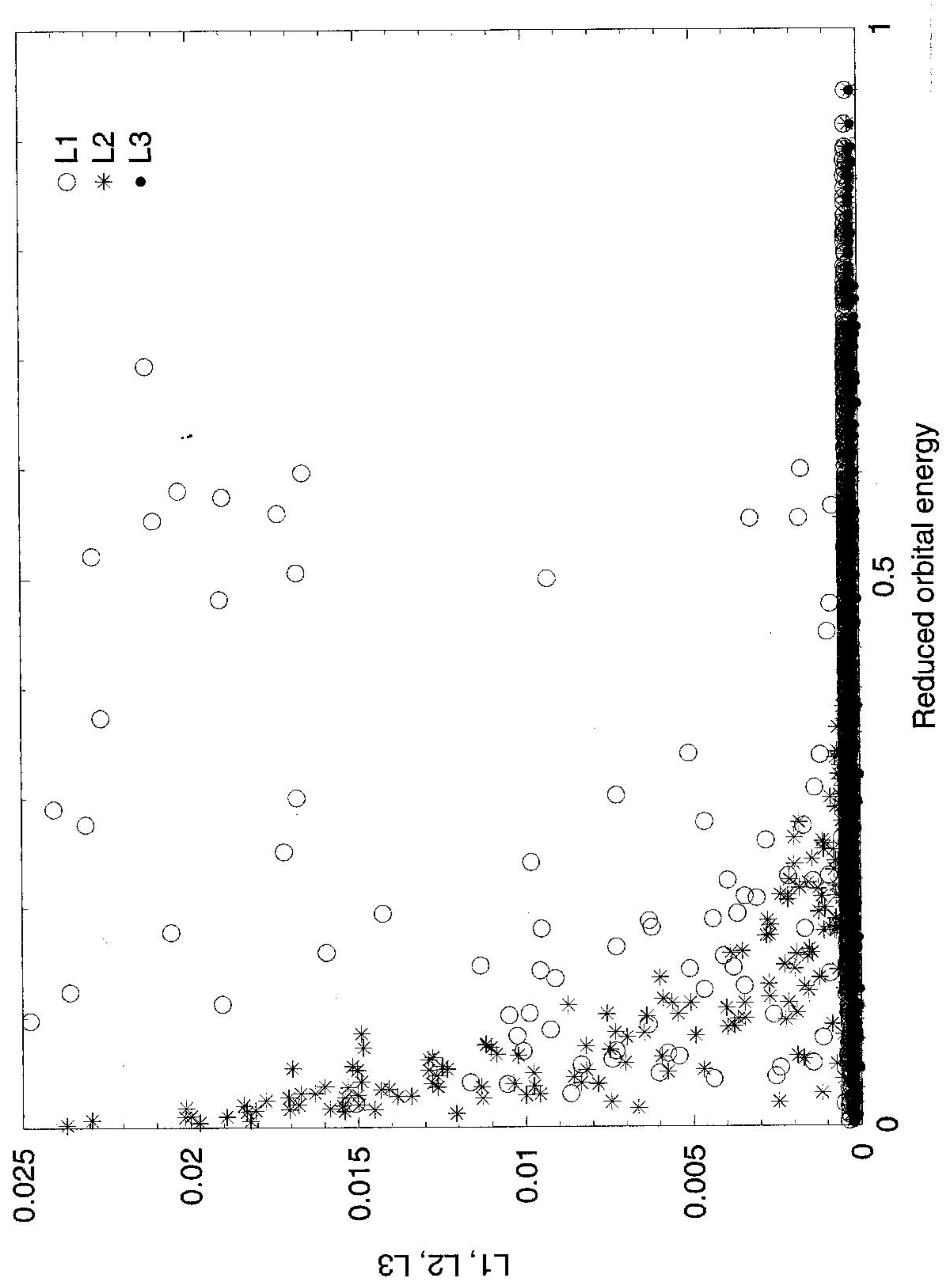

Fig. 2. Same as Figure 1, with the ordinate scale expanded to display more clearly the L2 values.

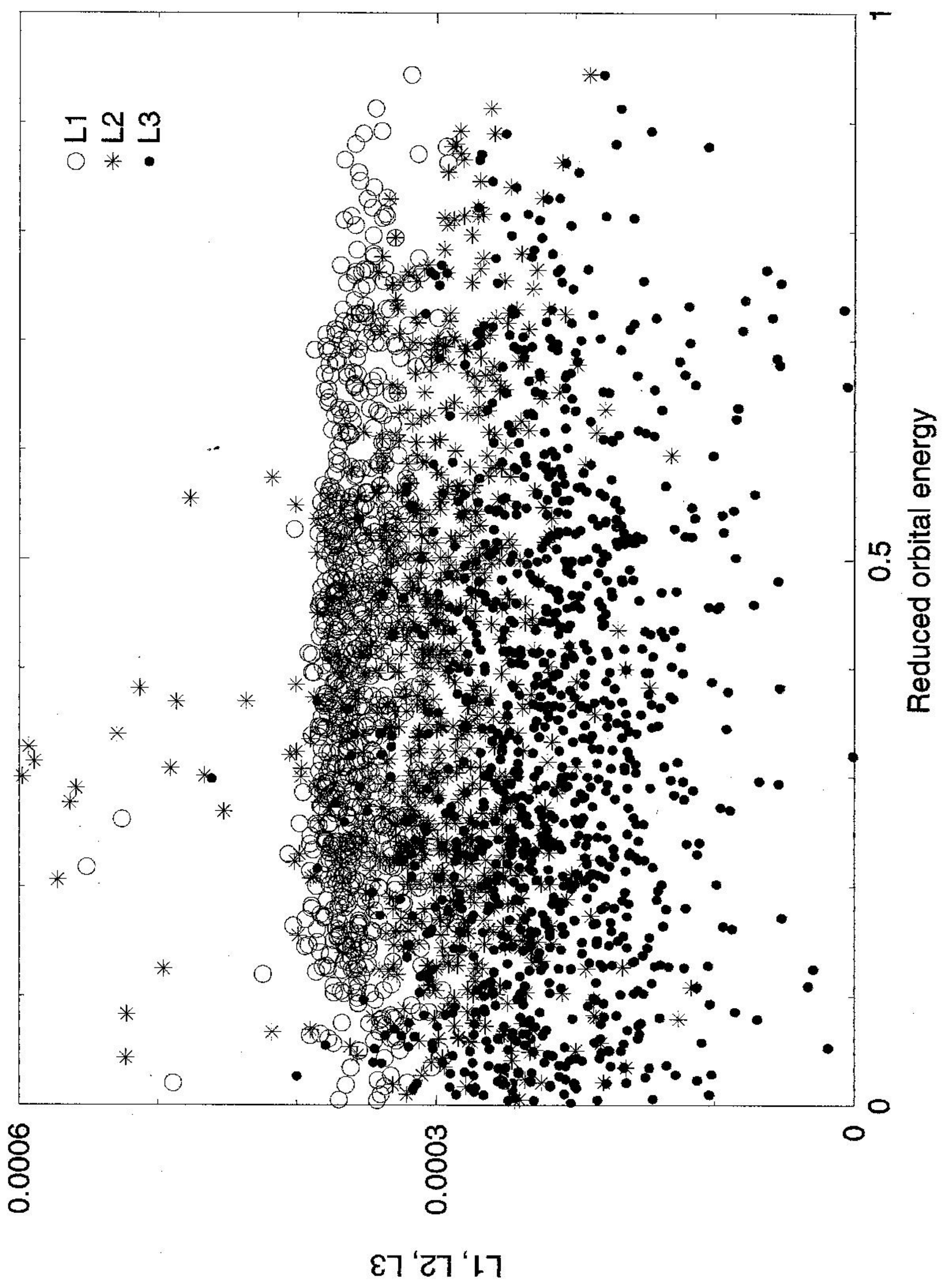

Fig. 3. Same as Figure 1, with the ordinate scale expanded to display more clearly the essentially zero values.

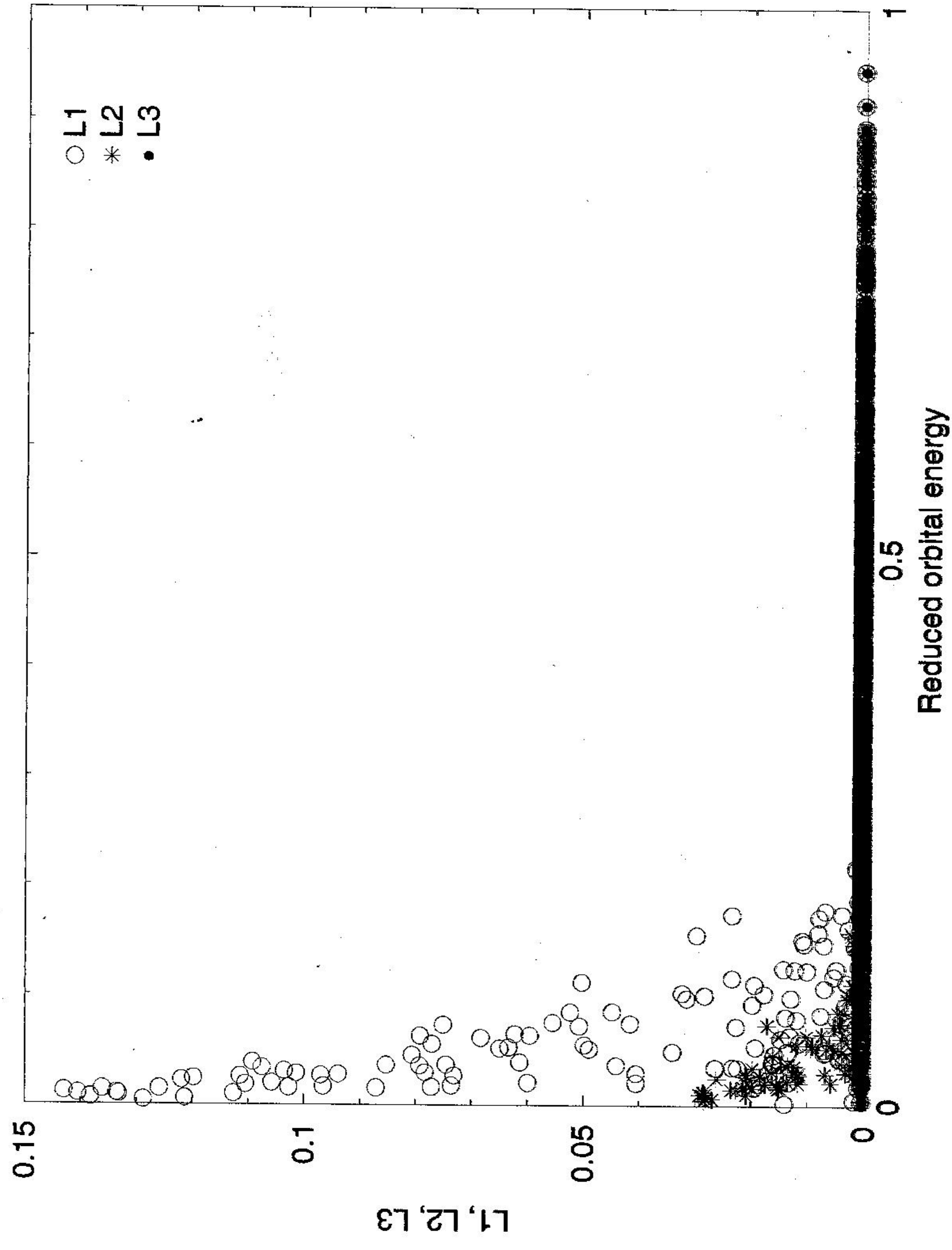

Fig. 4. Same as Figure 1, but the Liapunov exponents were obtained after artificially suppressing the Coriolis force.

lites to yield observational effects.

From a purely theoretical point of view, it is interesting how the value of the reduced energy allows us to distinguish regions where the motion may be chaotic, obeying only the Jacobi integral (low values), from those where it still may be chaotic, but obeying also a second integral or pseudo-integral (intermediate values), and those where it is fully regular, obeying three integrals or pseudo integrals (high values). The present result is, in a way, an extension to three dimensions of the classical result of Hénon and Heiles [1964] who, in a two dimensional potential, found that chaos set in for large energy values.

Our results also confirm the suggestion of Muzzio et al. [2000a] that the interaction of the Coriolis force and the attractive force of the cluster is the cause of the onset of chaos at intermediate reduced energy values. Nevertheless, even after artificially suppressing the Coriolis force, the Jacobi integral is the only isolating integral for reduced energies lower than 0.174 and there still remains a thin range of reduced energy (between 0.174 and 0.216) where a second integral may be present. Large reduced energy values correspond to orbits bound to the innermost regions of the satellite, where the centripetal-centrifugal force is weak. Therefore, it is reasonable to find only regular motions for those reduced energy values because the (fully integrable) attractive force from the satellite remains as the single relevant one in those regions after the Coriolis force is suppressed. Alternatively, orbits with low reduced energy values can reach large distances from the center of the satellite. In those outermost regions the centripetal-centrifugal force is comparable to the attractive force from the satellite (they become equal at the tidal radius) so that, in all likelihood, it is from the interplay between these two forces that the chaotic motions found at low reduced energy values arise. The influence of the Coriolis force is also present at those low values, but it seems to have a stabilizing effect, rather than favoring the onset of chaos as it does for intermediate values since, for low reduced energy values, the Liapunov exponents become higher when the Coriolis force is suppressed. This stabilizing effect may have some relationship to the well known fact that the Coriolis force helps to keep distant satellites bound [that is why planetary satellites can have much larger orbits if they have retrograde, rather than direct, motion; see, e.g., Innanen 1979].

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